Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2014 Autumn

Algebra

. Let $\phi : G \to G'$ be a group homomorphism with kernel K. Prove the following.

(a) $\phi(G)$, the image of G, is abelian if and only if $xyx^{-1}y^{-1} \in K$ for all $x, y \in G$.

(b) $\{x \in G \; : \; \phi(x) = \phi(a)\} = Ka$ for each $a \in G$.

2. Prove that each finite integral domain is a field.

3. Let R be a ring with multiplicative identity $\neq$, and let F be a field. Prove that if $\phi : R \to F$ is a surjective ring homomorphism, then the kernel of ϕ is a maximal ideal in R.

4. Let V be a vector space over the field F, and let $T : V \to V$ be a linear operator on V. Prove that

$$V_0 = \{\mathbf{v} \in V \mid T^k\mathbf{v} = \mathbf{0} \text{ for some integer } k \geq \quad \}$$

is a subspace of V, and if $T^m\mathbf{v} \in V_0$ for some $m \geq$, then $\mathbf{v} \in V_0$.

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Complex Analysis

. Find the image under the transformation

$$w = \frac{z - \ }{z + \ }$$

of (a) $\{z \in \mathbf{C} \quad |z+2| = \ \}$ and (b) the imaginary axis.

2. Use the method of residues to evaluate

$$ {}_{0}^{\infty} \frac{x^2\,dx}{(x^2 + \ )^3}.$$

3. Find the number of zeros, counting multiplicities, of

$$f(z) = z^6 - 5z^4 + z^3 - 2z$$

inside the circle $\{z \in \mathbf{C} \quad |z| = \ \}$, and justify your conclusion.

4. Find all Laurent series expansions of

$$f(z) = \frac{\ }{z(\ + z^3)}$$

centered at $z_0 = \ $ and their associated regions of convergence.

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Comprehensive Examination–Option III

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Applied Analysis

. Determine the solution $y = \phi$

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Numerical Analysis

. (a) Prove that there exist exactly two positive solutions of the equation

$$\ln x = (x-4)^2 - \quad .$$

(b) Find an approximation β of the smaller solution α such that $|\alpha - \beta| < \quad^{-6}$.

(c) Prove that your approximation β is in fact within $\quad^{-6}$ of (the exact) α.

Note: For this problem you may not use any graphing or rootfinding capabilities of your calculator.

2. Suppose that $f^{(5)}$ is continuous. Show that

$$f'''(x_0) = \frac{-f(x_0-2h) + 2f(x_0-h) - 2f(x_0+h) + f(x_0+2h)}{2h^3} \; + \; O(h^2).$$

3. Let A be a $n \times n$ band matrix of the following form.

$$\left(\begin{array}{cccccccc} 2 & & & & & & \cdots & \\ b_1 & 2 & & & & & \cdots & \\ c_1 & b_2 & 2 & & & & \cdots & \vdots \\ & c_2 & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & & \ddots & \ddots & \\ \vdots & \vdots & \ddots & c_{n-1} & & b_{n-2} & & \end{array} \right.$$

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Linear Programming

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Linear Programming–continued

4. Consider the following problem.

$$\begin{array}{rl} \text{Maximize} & 5x_1 + 8x_2 + 9x_3 \\ \text{Subject to} & 2x_1 + \ \ x_2 + \ \ x_3 \leq 2 \\ & 4x_1 + 2x_2 + 3x_3 \leq 3 \\ & x_1 + 3x_2 + 3x_3 \leq 4 \\ & x_1, x_2, x_3 \geq \end{array}$$

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Probability

. The negative binomial distribution is used to measure the number of Bernoulli trials one attempts until the rth success occurs. The probability mass function for the negative binomial is

$$P(X = x) =$$

Probability–continued

3. Suppose that we toss a fair coin until a head first comes up, and let X represent the number of tosses that were made. Then the possible values of X are , 2, ..., and the distribution function of X is defined by

$$m(i) = \frac{\quad}{2^i}$$

which is just the geometric distribution with parameter /2.

   (a) Find the expected value of X. Does this fit your intuition how it should be, given that the coin is fair?

   (b) Suppose that we flip a fair coin until a head first appears, and if the number of tosses equals n, then we are paid 2^n dollars. What is the expected value of the payment?

   (c) From what we learn in (b), how much would you be willing to pay per game for the privilege of playing this game?

4. A medical research team wishes to assess the usefulness of a certain symptom (S) in the diagnosis of a particular disease (D). In a random sample of 775 patients with the disease, 744 reported having the symptom. In an independent random sample of 38 subjects without the disease, 2 reported that they had the symptom.

   (a) Compute the sensitivity of the symptom, $P(S|D)$.

   (b) Compute the specificity of the symptom, $P(S^c|D^c)$, where ^{c} indicates the complement of the event.

   (c) Suppose it is known that the rate of the disease in the general population is . 2, $P(D) =$. 2.

      i. What is the positive predictive value of the symptom, $P(D|S)$?

      ii. What is the negative predictive value of the symptom, $P(D^c|S^c)$?

   (d) Find the positive predictive values for the symptom for the following disease rates: . , . , . .

   (e) What do you conclude about the positive predictive value of the symptom on the basis of the results obtained in part (d)?